CSCI 3110 Solution 4

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Problems below are from the text *Algorithms by Dasgupta, Papadimitriou, Vazirani*

1. Text 3.3 -See A3 solutions

2. Text 3.13

   (a) Prove that in any connected undirected graph $G = (V, E)$ there is a vertex $v \in V$ whose removal leaves G connected. (Hint: Consider the DFS search tree for G.)
   Consider the DFS tree of G starting at any vertex. If we remove a leaf (say v) from this tree, we still get a tree which is a connected subgraph of the graph obtained by removing v. Hence, the graph remains connected on removing v.

   (b) Give an example of a strongly connected directed graph $G = (V, E)$ such that, for every $v \in V$, removing v from G leaves a directed graph that is not strongly connected.
   A directed cycle. Removing any vertex from a cycle leaves a path which is not strongly connected.

   (c) In an undirected graph with 2 connected components it is always possible to make the graph connected by adding only one edge. Give an example of a directed graph with two strongly connected components such that no addition of one edge can make the graph strongly connected.
   A graph consisting of two disjoint cycles. Each cycle is individually a strongly connected component. However, adding just one edge is not enough as it (at most) allows us to go from one component to another but not back.

3. Text 3.15

3.15 (4 pts) The police department in the city of Computopia has made all streets one-way. The mayor contends that there is still a way to drive legally from any intersection in the city to any other intersection, but the opposition is not convinced. A computer program is needed to determine whether the mayor is right. However, the city elections are coming up soon, and there is just enough time to run a linear-time algorithm.

   (a) Formulate this problem graph-theoretically and explain why it can indeed be solved in linear time.

   This problem can be represented by a directed graph $G = (V, E)$ where each intersection is a vertex v and there is an edge between intersections (u, v) if there is a road that goes directly from u to v. The criteria that there is a way to drive legally from any intersection to any other intersection is true if, and only if, there is path from each vertex of G to every other vertex. This is true if, and only if, the graph is Strongly connected, so this can be solved in linear time with DFS by determining if the graph is a single strongly connected component.

   (b) Suppose it now turns out the mayor's original claim is false. She next claims something weaker: if you start driving from town hall, navigating one-wary streets, then no matter where you reach, there is always a way to drive legally back to the town hall. Formulate this weaker property as a graph-theoretic problem, and carefully show how it too can be checked in linear time.

We use the same graph G as in the first part of the question and label the town hall s. This property is true if, and only if, there is no path from s to a vertex v such that there is no path from v to s. Then there must be no path from s to a vertex outside of the connected component of s. To determine if the graph has this property we can label each vertex with the number of its connected component and then do a depth first search from s. If s can reach a vertex which is in a different connected component then the new claim is false.

4. Text 3.26

   (a) Show that an undirected graph has an Eulerian tour if and only if all its vertices have even degree. Conclude that there is no Eulerian tour of the Konigsberg bridges.
   Proof: (only if direction): Suppose an Eulerian tour exist. Consider a vertex that is visited (arrived at) k times via different edges as part of the tour (a cycle). Since the tour is Eulerian, each arrival edge must be paired with a distinct departure edge. Hence the degree of the vertex is even.
   (if direction) For the other direction we use induction on the number of vertices in the graph. First note that if $|V| = 2$, then trivially if the degree of both the vertices is even then the graph has an Eulerian tour. Let the statement be true for all graphs with $|V| = n$. We consider a graph G on $n + 1$ vertices such that all its vertices have even degrees. Let u be a vertex in this graph having neighbours $i_1, i_2, \dots i_{2k}$. Consider a graph G_1 where we remove u and add edges $(i_1, i_2), (i_3, i_4), \dots (i_{2k-1}, i_{2k})$ to G. Since G_1 has n vertices and the degree of each vertex is the same as in G (and thus even), G_1 must have an Eulerian tour. Replace every occurrence of the extra edges of the form (i_{t-1}, i_t) that we inserted, by (i_{t-1}, u) followed by (u, i_t). This gives an Eulerian tour of G.

   (b) To have an Eulerian path exactly two vertices (end points) must have odd degree and the rest must have even degree

   (c) In-degree for each vertex must be equal to Out-degree.

5. Text 4.4
   Work it out for yourself.

6. (4.5) Number of shortest paths
   We use an array to store the number of shortest paths to the vertices. The number of paths to a vertex v is the sum of the paths to its parents, through which the shortest path from the source s to v are the same and the shortest.
   We modify dijkstra algorithm to find the number of shortest paths. Since the edges have unit length, we can also modify BFS to achieve this goal.

```
DijkstraNumberOfPaths($G$)
for all $u \in V$
    dist($u$) = $\infty$
    paths($u$) = 0
dist($s$) = 0
paths($s$) = 1
$H$ = makequeue($V$)
while $H$ is not empty
    $u$ = deletemin($H$)
    for all edges $(u, v) \in E$:
        if dist($v$) > dist($u$) + $l(u, v)$
            dist($v$) = dist($u$) + $l(u, v)$
            paths($v$) = paths(u)
        else
            if dist($v$) == dist($u$) + $l(u, v)$
                paths($v$) = paths($v$) + paths($u$)
```

7. Text 4.8 - next assignment
8. Text 4.14 - next assignment